1 Sensitivity Analysis

Given a set of computed eigenvalues and eigenvectors it is sometimes useful to estimate the uncertainty in them due to uncertainties in the equations. Here we show how to estimate uncertainties due to the fact that elements of the matrices are represented by finite precision. These estimates will be developed for the nonlinear eigenvalue problem

$$\begin{aligned} \boldsymbol{A}(\lambda)\boldsymbol{x} &= \boldsymbol{0} \\ \boldsymbol{y}^*\boldsymbol{A}(\lambda) &= \boldsymbol{0} \end{aligned} \tag{1}$$

because it is the most general; any of the simpler problems can be written in the form of a nonlinear eigenvalue problem.

Define a scalar function of λ

$$f(\lambda) = \boldsymbol{y}^*\boldsymbol{A}(\lambda)\boldsymbol{x} \tag{2}$$

$$\frac{\partial f}{\partial \lambda} = \boldsymbol{y}^*\frac{\partial \boldsymbol{A}}{\partial \lambda}\boldsymbol{x} = \boldsymbol{y}^*\boldsymbol{A}'(\lambda)\boldsymbol{x} \tag{3}$$

If (a_{ij}) is the (i,j) element of $\boldsymbol{A}$ the partial of f (a_{ij}) is

$$\frac{\partial f}{\partial a_{ij}} = \bar{y}_i x_j \tag{4}$$

where $(\bar{\ })$ denotes the complex conjugate. The partial of λ with respect to that element then is

$$\frac{\partial \lambda}{\partial a_{ij}} = \frac{\partial f}{\partial a_{ij}}\left(\frac{\partial f}{\partial \lambda}\right)^{-1} = \frac{\bar{y}_i x_j}{\boldsymbol{y}^*\boldsymbol{A}'(\lambda)\boldsymbol{x}} \tag{5}$$

The precision of floating-point arithmetic on a computer can be characterized by a number known as *machine epsilon, defined as the smallest positive number* ϵ *such that* $1.0 - \epsilon < 1.0$*. In terms of* ϵ *the elements of the matrices are only known to lie in the interval*

$$[(1-\epsilon)a_{ij}, \quad (1+\epsilon)a_{ij}] \tag{6}$$

Due to this uncertainty in the (i,j) *element of matrix* $\boldsymbol{A}$*, the eigenvalue can only be said to lie within the interval* $\lambda \pm h_{ij}$ *where*

$$h_{ij} = \left| \frac{\bar{y}_i x_j}{\boldsymbol{y}^*\boldsymbol{A}'(\lambda)\boldsymbol{x}} \epsilon a_{ij} \right| \tag{7}$$

Summing the uncertainties in all matrix elements yields the total uncertainty in the eigenvalue due to uncertainties in $\boldsymbol{A}$

$$h = \sum_{i=1}^{n}\sum_{j=1}^{n} h_{ij} \tag{8}$$

Thus, due to the finite precision of the computer, the number of significant figures in an eigenvalue is $-\log_{10}(h\lambda)$ *The results of this analysis are displayed in the printed output by underlining the significant digits in the eigenvalues.*